IB GEOMETRY EXAMPLES 2

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Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `g.p.paternain@dpmms.cam.ac.uk`.

1. (a) Let $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ be a smooth function (U open) and suppose $Df|_p \neq 0$ for all $p \in f^{-1}(0)$. Show that $f^{-1}(0)$ is an orientable smooth surface in $\mathbb{R}^3$.

(b) Let Σ be an abstract smooth surface that can be covered by two charts (U_i, φ_i), $i = 1, 2$ with $U_1 \cap U_2$ connected. Show that Σ is orientable.

2. Show that each of the following parametrizations $\sigma : U \to \mathbb{R}^3$ is allowable, find the first fundamental form and sketch the image of σ.

(a) $U = \{(u, v) \in \mathbb{R}^2 \,|\, u > v\}$, $\quad \sigma(u, v) = (u + v, 2uv, u^2 + v^2)$;

(b) $U = \{(r, z) \in \mathbb{R}^2 \,|\, r > 0\}$, $\quad \sigma(r, z) = (r \cos(z), r \sin(z), z)$.

3. *Mercator's projection* of the sphere is the chart whose inverse is the local parametrization

$$\sigma(u, v) = (\operatorname{sech} u \cos v, \operatorname{sech} u \sin v, \tanh u).$$

Prove that this determines an allowable chart on the complement of a longitude, which sends lines of longitude and latitude to straight lines in the plane, and which preserves angles but not areas (*cf. Greenland versus Africa on a map; see also* `https://en.wikipedia.org/wiki/Mercator_projection`).

4. (a) Place the unit sphere $S^2 \subset \mathbb{R}^3$ inside a vertical circular cylinder C of radius one. Prove that horizontal projection from S^2 to C preserves area. Deduce that S^2 admits a smooth atlas of charts which are area-preserving.

(b) A *lune* is one component of the region on the unit sphere S^2 cut out by two great circles (so it is a bigon). Prove that if the lune has internal angle α, it has area 2α. Hence, or otherwise, prove that a *spherical triangle*, i.e. a connected region bound by 3 great circles and with internal angles α, β, γ each less than π, has area $\alpha + \beta + \gamma - \pi$.

5. (a) Let $\eta : (a, b) \to \mathbb{R}^3$ be a smooth curve given by $\eta(t) = (f(t), 0, g(t))$. Suppose η' is never zero, η is a homeomorphism to its image and $f(u) > 0$ for all u. Let Σ_η denote the associated surface of revolution given by rotating η around the z-axis. Prove that the Gauss curvature κ of Σ_η is given by

$$\kappa = \frac{(f'g'' - f''g')g'}{((f')^2 + (g')^2)^2 f}.$$

If η is parametrized by arc-length, show $\kappa = -f''/f$.

(b) Calculate κ for the hyperboloid of one sheet $\{x^2 + y^2 = z^2 + 1\}$ and of two sheets $\{x^2 + y^2 = z^2 - 1\}$. Describe the qualitative properties of κ (its sign, its behaviour near infinity). Illustrate the results with a picture.

6. Let Σ_η be as in the previous question. Let $n : \Sigma_\eta \to S^2$ be the Gauss map. Let $R_\theta : \mathbb{R}^3 \to \mathbb{R}^3$ denote rotation by angle θ about the z-axis. Prove that for $p \in \Sigma_\eta$, $n(R_\theta(p)) = R_\theta(n(p))$.

Hence, or otherwise, prove that for the hyperboloid $\{x^2 + y^2 = z^2 + 1\}$, the image of the Gauss map is the open annulus $\{|z| < 1/\sqrt{2}\} \subset S^2$.

7. Let $T \subset \mathbb{R}^3$ be the smooth embedded torus obtained by rotating the circle $(x - 2)^2 + z^2 = 1$ in the xz-plane around the z axis. Sketch T, and draw an illustration of where on T the Gauss curvature κ is positive, negative and zero.

8. Consider the surface $\Sigma \subset \mathbb{R}^3$ with parametrization

$$\sigma(u,v) = \gamma(u) + v\,a(u) \qquad u \in [0,2\pi), v \in (-1,1)$$

where $\gamma(u) = (\cos\, u, \sin\, u, 0)$ and $a(u) = (\cos(u/2)\cos(u), \cos(u/2)\sin(u), \sin(u/2))$. (This is an example of a 'ruled surface': one which is locally swept out by a moving Euclidean straight line.) Sketch Σ, and prove that Σ is a smooth surface for which the Gauss curvature κ is everywhere negative.

9. Let $\Sigma \subset \mathbb{R}^3$ be a smooth oriented surface, and $p \in \Sigma$. Let $n(p)$ be the unit normal vector at p. Let $v \in T_p\Sigma$ be a unit vector (with respect to the first fundamental form). Let γ_v be the plane curve which is the intersection of Σ and the affine two-plane $\mathbb{R}^2 = p + \mathrm{Span}\langle v, n(p)\rangle$. Viewing the second fundamental form as a bilinear form II_p on $T_p\Sigma$, show that $II_p(v,v)$ is the curvature of the plane curve γ_v at p.

[*The curvature of a plane curve* $\gamma : (a,b) \to \mathbb{R}^2$ *parametrized by arc-length is the function* $\kappa : (a,b) \to \mathbb{R}$ *for which* $\gamma''(s) = \kappa(s)n_\gamma(s)$*, with* $n_\gamma(s)$ *the unit normal to* γ *for which* $\langle \gamma'(s), n_\gamma(s)\rangle$ *forms a positively oriented basis for* $\mathbb{R}^2$.]

10. The *tractrix* is the path followed by a heavy object which starts at $(1,0)$ in $\mathbb{R}^2$ and is pulled by a person attached to the object by a (taut) rope of length 1 and who walks from the origin up the y-axis. The *tractroid* (or pseudosphere) is the surface obtained by rotating the tractrix around the y-axis. Show that the tractrix can be described parametrically as

$$x = \sin t,\ \ y = (\cos t + \log\tan(t/2)), \qquad t \in (\pi/2, \pi).$$

Prove that the tractroid is a smooth surface where $y > 0$, has Gauss curvature identically -1, and has total area 2π.

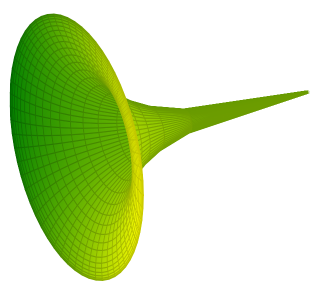

FIGURE 1. The tractroid