Algebraic Geometry, Part II, Example Sheet 4,2012

Assume throughout that the base field k is algebraically closed. If it helps, feel free to assume throughout that it has characteristic zero.

1. Let V be a smooth irreducible projective curve and $P \in V$ any point. Show that there exists a nonconstant rational function on V which is regular everywhere except at P. Show moreover that there exists an embedding $\phi\colon V \hookrightarrow \mathbb{P}^n$ such that $\phi^{-1}(\{X_0 = 0\}) = \{P\}$. In particular, $V \setminus \{P\}$ is an affine curve. If V has genus g, show that there exists a nonconstant morphism $V \to \mathbb{P}^1$ of degree g.

2. Let P_∞ be a point on an elliptic curve X (smooth irreducible projective curve of genus 1) and $\alpha_{3P_\infty}\colon X \xrightarrow{\sim} W \subset \mathbb{P}^2$ the projective embedding, with image W. Show that $P \in W$ is a point of inflection if and only if $3P = 0$ in the group law determined by P_∞. Deduce that if P and Q are points of inflection then so is the third point of intersection of the line PQ with W.

3. Let $V:\ ZY^2 + Z^2Y = X^3 - XZ^2$ and take $P_0 = (0:1:0)$ for the identity of the group law. Calculate the multiples $nP = P \oplus \cdots \oplus P$ of $P = (0:0:1)$ for $2 \le n \le 4$.

4. Show that any morphism from a smooth irreducible projective curve of genus 4 to a smooth irreducible projective curve of genus 3 must be constant.

5. (Assume $\mathrm{char}(k) \neq 2$) (i) Let $\pi\colon V \to \mathbb{P}^1$ be a hyperelliptic cover, and $P \neq Q$ ramification points of π. Show that $P - Q \not\sim 0$ but $2(P-Q) \sim 0$.

   (ii) Let $g(V) = 2$. Show that every divisor of degree 2 on V is linearly equivalent to $P + Q$ for some $P, Q \in V$, and deduce that every divisor of degree 0 is linearly equivalent to $P - Q'$ for some P, $Q' \in V$.

   (iii) Show that if $g(V) = 2$ then the subgroup $\{\, [D] \in \mathrm{Cl}^0(V) \mid 2[D] = 0 \,\}$ of the divisor class group of V has order 16.

6. Show that a smooth plane quartic is never hyperelliptic.

7. Let $V:\ X_0^6 + X_1^6 + X_2^6 = 0$, a smooth irreducible plane curve. By applying the Riemann–Hurwitz formula to the projection to $\mathbb{P}^1$ given by $(X_0 : X_1)$, calculate the genus of V.

   Now let $\phi\colon V \to \mathbb{P}^2$ be the morphism $(X_i) \mapsto (X_i^2)$. Identify the image of ϕ and compute the degree of ϕ.

8. Let $V \subset \mathbb{P}^3$ be the intersection of the quadrics $Z(F)$, $Z(G)$ where $\mathrm{char}(k) = 0$ and

$$F = X_0X_1 + X_2^2, \quad G = \sum_{i=0}^{3} X_i^2$$

   (i) Show that V is a smooth curve (possibly reducible).

   (ii) Let $\phi = (X_0\ X_1\ X_2)\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$. (This map is the projection from the point $(\,0\ 0\ 0\ 1)$ to $\mathbb{P}^2$.) Show that $\phi(V)$ is a conic $C \subset \mathbb{P}^2$. By parametrising C, compute the ramification of ϕ and show that $\phi\colon V \to C$ has degree 2. Deduce that V is irreducible of genus 1.

9. In this example, for any set of six points $\{P_i\}$ in $\mathbb{P}^1$ we construct a smooth curve of genus 2 in $\mathbb{P}^3$, together with a morphism of degree 2 branched precisely at $\{P_i\}$. Assume $\operatorname{char}(k) \neq 2$ thoughout.

   (i) Show that coordinates on $\mathbb{P}^1$ may be chosen for which the points P_i are $0, \infty$ and the roots of 2 coprime quadratic polynomials $p(x) = x^2 + ax + b$, $q(x) = x^2 + cx + d$, with $bd \neq 0$.

   (ii) Let $C \subset \mathbb{A}^2$ be the affine curve with equation $y^2 = h(x)$ where $h(x) = xp(x)q(x)$. Show that C is nonsingular, and that $\pi\colon C \to \mathbb{A}^1$, $(x, y) \mapsto x$ is 2-to-1 except at points of the form $P = (x, 0)$, at which $e_P = 2$.

   (iii) Let $W = V(\{F, G\}) \subset \mathbb{P}^3$ be the projective variety given by

   $$F(\underline{X}) = X_2^2 X_0 - X_1(X_3 + aX_1 + bX_0)(X_3 + cX_1 + dX_0), \quad G(\underline{X}) = X_0X_3 - X_1^2$$

   Show that the affine piece $W \cap \{X_0 \neq 0\}$ is isomorphic to C, but that $W \cap \{X_0 = 0\}$ is a line. In particular, W is reducible.

   (iv) Let $F'(\underline{X}) = X_1X_2^2 - X_3(X_3 + aX_1 + bX_0)(X_3 + cX_1 + dX_0)$. Show that $X_0F' \in I^h(W)$. Let $V = V(\{F, F', G\}$. Show that $V \cap \{X_0 \neq 0\} = W \cap \{X_0 \neq 0\}$. Show also that $V \cap \{X_0 = 0\}$ is a single point, and that it is a smooth point of V.

   (v) Deduce that V is a smooth irreducible projective curve of genus 2, and that the morphism $\pi = (X_0\ X_1)\colon V \to \mathbb{P}^1$ has degree 2.

10. (i) Let V be a smooth irreducible projective curve of genus $g \geq 2$. Observe that for $P \in V$ the Riemann–Roch theorem implies that $\ell(mP) \geq 1 - g + m$. We say that P is a *Weierstrass point* of V is $\ell(gP) \geq 2$. Show that if $g = 2$, the Weierstrass points of V are the ramification points of the hyperelliptic morphism $\pi\colon V \to \mathbb{P}^1$.

    (ii) Prove that for any hyperelliptic curve V the ramification points of $\pi\colon V \to \mathbb{P}^1$ are Weierstrass points.

    (iii) Let V be a smooth plane quartic. Show that $P \in V$ is a Weiersrtrass point if and only if it is a point of inflexion.