Part II Algebraic Geometry

Example Sheet II, 2015
(For all questions, assume k is algebraically closed.)

1. Given distinct points $P_0, \cdots, P_{n+1}$ in $\mathbf{P}^n$, no $(n+1)$ of which are contained in a hyperplane, show that homogeneous coordinates may be chosen on $\mathbf{P}^n$ so that $P_0 = (1\colon 0\colon \ldots\colon 0)$, $\cdots$, $P_n = (0\colon \ldots\colon 0\colon 1)$ and $P_{n+1} = (1\colon 1\colon \ldots\colon 1)$. [This generalises to arbitrary n a result you are very familiar with when $n = 1$.]

2. Given hyperplanes $H_0, \cdots, H_n$ of $\mathbf{P}^n$ such that $H_0 \cap \cdots \cap H_n = \emptyset$, show that homogeneous coordinates $x_0, \ldots, x_n$ can be chosen on $\mathbf{P}^n$ such that each H_i is defined by $x_i = 0$.

3. Let W be an n-dimensional vector space over k. Denote by $\mathbf{P}(W)$ the projective space $(W \setminus \{0\})/\sim$, where the equivalence relation is the usual rescaling. Show that the set of hyperplanes in $\mathbf{P}(W)$ is parametrized by $\mathbf{P}(W^*)$, where W^* is the dual vector space to W. If $P_1, \cdots, P_N$ are points of $\mathbf{P}(W)$, describe the set in $\mathbf{P}(W^*)$ corresponding to hyperplanes not containing any of the P_i. Deduce (using k infinite) that there are infinitely many such hyperplanes.

4. Let V be a hypersurface in $\mathbf{P}^n$ defined by a non-constant homogeneous polynomial F, and L a (projective) line in $\mathbf{P}^n$; show that V and L must intersect in a non-empty set.

5. Let X be an algebraic set (in affine or projective space), and suppose that $X = X_1 \cup \cdots \cup X_n$ and $X = X_1' \cup \cdots \cup X_m'$ are two decompositions into irreducible components, such that $X_i \not\subseteq X_j$ for any $i \neq j$, and $X_i' \not\subseteq X_j'$ for any $i \neq j$. Show that $n = m$ and after reordering, $X_i = X_i'$. Thus irreducible decompositions are essentially unique.

6. Decompose the algebraic set V in $\mathbf{P}^3$ defined by equations $x_2^2 = x_1x_3$, $x_0x_3^2 = x_2^3$ into irreducible components.

7. Assume $\operatorname{char} k \neq 2$.

   i) Show that a homogeneous polynomial $F(x_0, x_1, x_2)$ of degree 2 can be written uniquely in the form $\mathbf{x}^T A\mathbf{x}$, where A is a 3×3 symmetric matrix with entries in k and $\mathbf{x}^T = (x_0, x_1, x_2)$; show that the polynomial is irreducible if and only if $\det(A) \neq 0$. Let $V \subset \mathbf{P}^2$ be the algebraic set defined by the equation $F = 0$; if V is irreducible and k algebraically closed, show that you can choose coordinates such that $F = x_0^2 + x_1^2 + x_2^2$, and that V is isomorphic to $\mathbf{P}^1$.

   ii) In contrast, show that if $f(x, y) \in k[x, y]$ is an irreducible (non-homogeneous!) polynomial of degree 2, k algebraically closed, then $Z(f)$ is either $\mathbf{A}^1$ or $\mathbf{A}^1 \setminus \{0\}$.

8. Consider the projective plane curves corresponding to the following affine curves in $\mathbf{A}^2$.

(a) $y = x^3$	(b) $xy = x^6 + y^6$
(c) $x^3 = y^2 + x^4 + y^4$	(d) $x^2y + xy^2 = x^4 + y^4$
(e) $2x^2y^2 = y^2 + x^2$	(f) $y^2 = f(x)$ with f a polynomial of degree n.

   In each case, calculate the points at infinity of these curves, i.e., homogenize the equations to obtain equations for a curve in $\mathbf{P}^2$ and identify the resulting points at infinity. Furthermore, find the singular points of the projective curve. If you wish, you may make assumptiosn about the characteristic of k to simplify the analysis.

9. If $F(x_0, \ldots, x_n)$ a homogeneous polynomial of degree $d > 0$, prove that $dF = \sum_{i=0}^{n} x_i\, \partial F/\partial x_i$. If F is irreducible, let $X = Z(F) \subset \mathbf{P}^n$ be the projective variety defined by $F = 0$. We say a point $p \in X$ is a *non-singular point* of X if, denoting by $U_0, \ldots, U_n$ the standard cover of $\mathbf{P}^n$ by affine spaces, p is a non-singular point of $X \cap U_i$ whenever $p \in X \cap U_i$. Show that the singular locus of X (the set of points of X which are not non-singular) consists precisely of the points p in $\mathbf{P}^n$ with $\partial F/\partial x_i(p) = 0$ for $i = 0, \ldots, n$.