ALGEBRAIC GEOMETRY, SHEET II: LENT 2020

Throughout this sheet, the symbol k will denote an algebraically closed field.

1. Let $X \subset \mathbb{P}^2$ be a projective variety. A *morphism* $\mathbb{P}^1 \to X$ is a polynomial map
$$\varphi([X_0 : X_1]) = (f_0([X_0 : X_1]), f_1([X_0 : X_1]), f_2[(X_0 : X_1]))$$
where f_i are homogeneous of the same degree, such that $f(\mathbb{P}^1) \subset X$. For any irreducible conic $C \subset \mathbb{P}^2$, there is a bijective morphism
$$\varphi : \mathbb{P}^1 \to C.$$
2. Let $X = \mathbb{V}(F)$ be a hypersurface in $\mathbb{P}^n$ and let ℓ be a line, i.e. a subvariety defined by $n-1$ linearly independent linear homogeneous polynomials. Show that X intersects ℓ in a nonempty set.
3. A *quasi-projective variety* X (or simply a variety in this course) is a complement $Y \setminus W$ where Y is a projective variety and $W \subset Y$ is a (possibly empty) closed subvariety. Give examples of quasi-projective varieties that are neither affine nor projective. Prove that any quasi-projective variety X is covered by affine varieties.
4. Consider the morphism $\mathbb{A}^2 \to \mathbb{A}^2$ sending (x, y) to (x, xy). Prove that the image of this morphism is not quasi-projective.
5. Let $I \subset k[z_1, \ldots z_n]$ be an ideal. Let $f_1, \ldots, f_r$ be generators for this ideal. Prove or give a counterexample to the following statement relating homogenizations of polynomials and ideals:
$$I^h = \langle f_1^h, \ldots, f_r^h \rangle \subset k[Z_0, \ldots, Z_n].$$
6. A collection of points in $\mathbb{P}^n$ is said to be in *general position* if no subset of $n+1$ or fewer points is linear dependent (as vectors in k^{n+1}). If S is a collection of $2n$ points in $\mathbb{P}^n$, prove that S is the zero locus of a collection of quadratic polynomials.
7. Recall that the *Segre surface* $\Sigma_{1,1} \subset \mathbb{P}^3$ is given by $\mathbb{V}(Z_0Z_3 - Z_1Z_2)$. Calculate the field of rational functions of $\Sigma_{1,1}$. Describe the set of all lines contained on this surface.
8. Consider the cubic surface $S \subset \mathbb{P}^3$ given by $\mathbb{V}(Z_0^3 + \ldots + Z_3^3)$. Find a line ℓ contained on this surface[1]. Consider the collection of planes in $\mathbb{P}^3$ passing through this ℓ and describe the corresponding collection of equations for these planes. Given such a plane H containing ℓ, describe the curve $H \cap S$. Given a general plane in $\mathbb{P}^3$ (in particular, not necessarily containing ℓ) describe its intersection with S.
9. Construct two non-isomorphic irreducible cubic plane curves C_1 and C_2 in $\mathbb{P}^2$, such that the fraction field of the coordinate rings of C_1 and C_2 are both isomorphic to $k(z)$. Draw appropriate pictures.

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[1]This is part of a famous geometry. A (smooth) cubic surface contains exactly 27 lines no matter what the equation is. Can you find the 27 lines in this case?

10. *(Veronese varieties)* Let $\{F_I\}$ be the set of degree d monomials in $n+1$ variables $Z_0, \ldots, Z_n$. Consider the map
$$\nu_d : \mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}-1}$$
sending a tuple $[Z_0 : \cdots : Z_n]$ to $[\cdots : F_I : \cdots]$, i.e. to the tuple of monomials of degree d. Find generators for the image of ν_d and prove that ν_d is an isomorphism onto its image.
11. Write down the projective closures of the following curves, determine the points at infinity, and find all singular points:
    - $xy = x^6 + y^6$.
    - $x^3 = y^2 + x^4 + y^4$
    - $y^2 = f(x)$ with $f(x)$ a polynomial of degree d.
12. *(Dual projective space)* Let V be a finite dimensional vector space. Prove that the set of hyperplanes in $\mathbb{P}(V)$ is naturally isomorphic to $\mathbb{P}(V^\star)$, where $V^\star$ is the dual vector space. Using the fact that two distinct lines intersect at a unique point it $\mathbb{P}^2$, deduce the fact that there is a unique line through two distinct points in $\mathbb{P}^2$.
13. *(Grassmannian)* In this exercise, you will contemplate an important generalization of projective space known as the Grassmannian. Let V be an n-dimensional vector space and $0 \le k \le n$ an integer. Let $G(k, V)$ be the set of k-dimensional linear subspaces of V.
    (a) Consider k linearly independent vectors $v_1, \ldots, v_k$ in V and choose a basis to represent them as a $k \times n$ matrix M. Observe that $GL(k)$ acts on the set of such matrices by left multiplication without affecting the associated vector space. Prove that the $k \times k$ minors of such a matrix give rise to a well-defined map
    $$\iota : G(k, V) \to \mathbb{P}^{\binom{n}{k}-1}.$$
    (b) Prove that ι is injective.
    (c) ($\star$) Prove that the image of ι is Zariski closed. (Hint: Given a subspace W represented by a matrix M_W, you may assume that the first $k \times k$ block of M_W is the identity. The rest of M_W is a $k \times (n-k)$ matrix A. How are the maximal minors of M_W related to the minors of A? The minors of A satsify relations coming from Laplace expansion. This gives you equations on an affine patch.)
14. Consider C a smooth curve given by the projective closure of an affine curve of the form $y^2 = f(x)$ over the complex numbers. Construct a morphism
$$\pi : C \to \mathbb{P}^1$$
such that at all but finitely many points q of $\mathbb{P}^1$, the preimage $\pi^{-1}(q)$ consists of exactly two points[2].
15. ($\star$) Let $d \ge 1$ be an integer and $C \subset \mathbb{P}^2_{\mathbb{C}}$ the union of a collection of d lines. Assume the lines are chosen generically. In the Euclidean topology in $\mathbb{P}^2$, describe the topological space of C and draw a picture of it. Let f_d be a degree d homogeneous polynomial

[2]Given a morphism $C \to \mathbb{P}^1$, the number of preimages of a generic point on $\mathbb{P}^1$ is the *degree* of the map, which in this case will be 2.

in 3 variables, and assume the coefficients of f_d are chosen generically. Based on your analysis above, can you guess a description of the Euclidean topological space $\mathbb{V}(f_d)$? It may help you to carefully compare $\mathbb{V}(Z_0Z_1)$ and $\mathbb{V}(Z_0Z_1 - Z_2^2)$.