ALGEBRAIC GEOMETRY, SHEET III: LENT 2022

1. Let $V_1 = \mathbb{V}(X_0^8 + X_1^8 + X_2^8)$ and $V_2 = \mathbb{V}(X_0^4 + X_1^4 + X_2^4)$ be curves in $\mathbb{P}^2$. Show that each is smooth and irreducible.
2. Let V_1 and V_2 be as in the previous question. Let $\varphi : V_1 \to V_2$ be the morphism sending X_i to X_i^2. Determine the degree of this morphism and compute the ramification indices for all points of V_1.
3. Let $V \subset \mathbb{P}^2$ be a smooth degree d curve in $\mathbb{P}^2$. Let P be a point on V and consider the projection from P restricted to V:
$$\pi : V \to \mathbb{P}^1.$$
Prove that for all but finitely many points Q on $\mathbb{P}^1$, the set $\pi^{-1}(Q)$ contains $d-1$ points. Deduce the degree of π.
4. We have seen in lecture that if ω and ω' are differentials on a curve V, then $\mathrm{div}(\omega)$ and $\mathrm{div}(\omega')$ differ by a principal divisor. Let t be the coordinate on the first affine patch $\mathbb{A}^1 \subset \mathbb{P}^1$. Explicitly calculate the divisor of $d(t^2)$, and thereby verify that it is equivalent to the divisor of dt.
5. Let U be the complement of the origin in $\mathbb{A}^2$. Consider a rational map
$$\varphi : \mathbb{A}^2 \dashrightarrow \mathbb{A}^1$$
that is regular on all points of U. Prove that φ is a morphism. Comment on whether U can be an affine variety.
6. Let $A = \mathbb{C}[X_0, X_1, Y_0, Y_1]$. An element of A is bihomogeneous of bidegree (d, e) if it is homogeneous of degree d in the X_i variables, treating the Y_i as scalars, and is similarly homogeneous of degree e in the Y_i variables. Let f in A be bihomogeneous of bidegree (d, e). Prove[1] that the vanishing locus of f is a well-defined subset of $\mathbb{P}^1 \times \mathbb{P}^1$.
7. Let $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ be the Segre embedding. Let $V \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a Zariski closed subset in the subspace topology on $\mathbb{P}^1 \times \mathbb{P}^1$. Prove that V is the vanishing locus of a set of bihomogeneous polynomials.
8. Let $V \subset \mathbb{P}^1 \times \mathbb{P}^1$ be the vanishing locus of a bihomogeneous polynomial. Prove that V is closed in the subspace topology under the embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$.
9. It is true that every pair of curves in $\mathbb{P}^2$ intersect. Now consider a morphism $\varphi : \mathbb{P}^2 \to \mathbb{P}^1$. By examining sets of the form $V_Q = \varphi^{-1}(Q)$ for different Q, prove that φ must be constant. Deduce that $\mathbb{P}^2$ is not isomorphic to a subvariety of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

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[1]There is nothing in this question that is meant to trick you.