Part II Algebraic geometry

Example Sheet I, 2024

In all problems, you may assume that we are working over an algebraically closed field k. While there is a little development of theory on this example sheet, the main point is to be exposed to lots of different examples for which there is not time to cover in lecture. Getting a feeling for Algebraic Geometry often means not just absorbing the definitions but also seeing and working with many examples. Questions marked with a $*$ should be viewed as more challenging.

1. Let $X \subseteq \mathbf{A}^n$ be an arbitrary subset. Show that $Z(I(X))$ coincides with the closure of X, i.e., the smallest Zariski closed subset of $\mathbf{A}^n$ containing X.

2. Show that any non-empty open subset of an irreducible algebraic set (i.e., a variety) is dense and irreducible. Show that if an affine variety is Hausdorff, it consists of a single point.

3. Recall a *basis* for a set X is a collection $\mathcal{U}$ of subsets of X such that (1) for every $x \subseteq X$, there is a $U \in \mathcal{U}$ with $x \in U$ and (2) for every $U_1, U_2 \in \mathcal{U}$ and $x \in U_1 \cap U_2$, there is a $U_3 \in \mathcal{U}$ such that $x \in U_3 \subseteq U_1 \cap U_2$. A basis defines a topology on X, with open subsets of X being arbitrary unions of sets in $\mathcal{U}$. Show that if X is an affine variety, then the collection of open sets $\{X \setminus Z(f) \,|\, f \in A(X)\}$ forms a basis for the topology of X.

4. A topological space is called *Noetherian* if it satisfies the descending chain condition for closed subsets. Show that affine varieties are Noetherian in the Zariski topology.

5. Let X be an algebraic set in affine, and suppose that $X = X_1 \cup \cdots \cup X_n$ and $X = X_1' \cup \cdots \cup X_m'$ are two decompositions into irreducible components, such that $X_i \not\subseteq X_j$ for any $i \neq j$, and $X_i' \not\subseteq X_j'$ for any $i \neq j$. Show that $n = m$ and after reordering, $X_i = X_i'$. Thus irreducible decompositions are essentially unique.

6. Let $Y \subseteq \mathbf{A}^2$ be the curve given by $xy = 1$. Show that Y is not isomorphic to $\mathbf{A}^1$. Find all morphisms $\mathbf{A}^1 \to Y$ and $Y \to \mathbf{A}^1$.

7. Let $Y \subseteq \mathbf{A}^3$ be the set $\{(t, t^2, t^3) \,|\, t \in k\}$. Show that Y is an affine variety, determine $I(Y)$, and show that $A(Y)$ is a polynomial ring in one variable. Y is called the *twisted cubic*.

8. Let $Y = Z(x^2 - yz, xz - x)$. Show that Y has 3 irreducible components. Describe them, and their corresponding prime ideals.

9. Show that if $X \subseteq \mathbf{A}^n$, $Y \subseteq \mathbf{A}^m$ are affine varieties, then $X \times Y \subseteq \mathbf{A}^n \times \mathbf{A}^m = \mathbf{A}^{n+m}$ is a Zariski closed subset of $\mathbf{A}^{n+m}$. The more ambitious may try to show that $X \times Y$ is irreducible, but this is hard!

10*. Let $Y \subseteq \mathbf{A}^3$ be the set $\{(t^3, t^4, t^5) \mid t \in k\}$. Show that Y is an affine variety, and determine $I(Y)$. Show $I(Y)$ cannot be generated by two elements.

11*. Suppose the characteristic of k is not 2. Show that there are no non-constant morphisms from $\mathbf{A}^1$ to $E = Z(y^2 - x^3 + x) \subseteq \mathbf{A}^2$. [Hint: Consider the map $A(E) \to A(\mathbf{A}^1) = k[t]$, and the images of x and y under this map. Then use the fact that $k[t]$ is a UFD.]

12. Let $f \in k[x_1, \ldots, x_n]$ be an irreducible polynomial, and consider $Y = Z(yf - 1) \subseteq \mathbf{A}^{n+1}$, with coordinates $x_1, \ldots, x_n, y$. Show that Y is irreducible. Show that the projection $\mathbf{A}^{n+1} \to \mathbf{A}^n$ given by $(x_1, \ldots, x_n, y) \mapsto (x_1, \ldots, x_n)$ induces a morphism $Y \to \mathbf{A}^n$ which is a homeomorphism to its image $D(f) := \{(a_1, \ldots, a_n) \in \mathbf{A}^n \mid f(a_1, \ldots, a_n) \neq 0\}$. This gives the Zariski open set $D(f)$ the structure of an algebraic variety.

13*. Let $f, g \in k[x, y]$ be polynomials, and suppose f and g have no common factor. Show there exists $u, v \in k[x, y]$ such that $uf + vg$ is a non-zero polynomial in $k[x]$.

Now let $f \in k[x, y]$ be irreducible. The variety $Z(f)$ is called an affine *plane curve*. Show that any proper subvariety of $Z(f)$ is finite.

14. Let A be a k-algebra. We say A is *finitely generated* if there is a surjective k-algebra homomorphism $k[x_1, \ldots, x_n] \to A$ for some n. Now suppose that A is a finitely generated k-algebra which is also an integral domain. Show that there is an affine variety Y with A isomorphic to $A(Y)$ as k-algebras.

15. Let $X = Z(x_1x_2 - x_3x_4) \subseteq \mathbf{A}^4$. Let $U, V \subseteq X$ be the open sets defined by $U := X \setminus Z(x_1, x_3)$ and $V := X \setminus Z(x_1)$. Thus $\varphi := x_4/x_1$ defines a regular function on V. Show that φ extends to a regular function on U. Having done this, meditate on why this shows the definition of regular function given in lecture needs to be as complicated as it is.