Part II Algebraic Geometry

Example Sheet IV, 2024

(For all questions, assume k is algebraically closed. Further, you can assume the characteristic is not equal to 2 if necessary. A * indicates a more difficult problem. I should note in general that this example sheet is a fair bit harder than the previous ones, so don't despair if you don't get all the problems!)

1. If P is a smooth point of an irreducible curve X and $t \in \mathcal{O}_{X,P}$ is a local parameter at P, show that $\dim_k \mathcal{O}_{X,P}/(t^n) = n$ for every $n \in \mathbb{N}$.

2. Show that $X = Z(x_0^8 + x_1^8 + x_2^8)$ and $Y = Z(y_0^4 + y_1^4 + y_2^4)$ are irreducible smooth curves in $\mathbb{P}^2$ provided $\mathrm{char}(k) \neq 2$, and that $\phi\colon (x_i) \mapsto (x_i^2)$ is a morphism from X to Y. Determine the degree of ϕ, and compute e_P for all $P \in X$.

3. Show that the plane cubic $X = Z(f)$, $f = x_0x_2^2 - x_1^3 + 3x_1x_0^2$, is smooth if $\mathrm{char}(k) \neq 2, 3$. Find the degree and ramification degrees (i.e., the e_P) for (i) the projection $\phi = (x_0 : x_1)\colon X \to \mathbb{P}^1$ (ii) the projection $\phi = (x_0 : x_2)\colon X \to \mathbb{P}^1$.

4. Let X be a non-singular projective curve. Let $V \subset K(X)$ be a finite-dimensional k-vector subspace of $K(X)$. Show that there exists a divisor D on X for which $V \subset \mathcal{L}(D)$.

5. Let X be a smooth plane cubic. Assume that V has equation
$$x_0x_2^2 = x_1(x_1 - x_0)(x_1 - \lambda x_0),$$
for some $\lambda \in k \setminus \{0, 1\}$.

   Let $P = (0 : 0 : 1)$ be the point at infinity in this equation. Writing $x = x_1/x_0$, $y = x_2/x_0$, show that x/y is a local parameter at P. [Hint: consider the affine piece $x_2 \neq 0$.] Hence compute $v_P(x)$ and $v_P(y)$. Show that for each $m \geq 1$, the space $\mathcal{L}(mP)$ has a basis consisting of functions x^i, x^jy, for suitable i and j, and that $\ell(mP) = m$.

6. Let $f \in k[x]$ a polynomial of degree $d > 1$ with distinct roots, and $V \subset \mathbb{P}^2$ the projective closure of the affine curve with equation $y^{d-1} = f(x)$. Assume that $\mathrm{char}(k)$ does not divide $d - 1$. Prove that V is smooth, and has a single point P at infinity. Calculate $v_P(x)$ and $v_P(y)$.

   * Deduce (without using Riemann–Roch) that if $n > d(d-3)$, then $\ell((n+1)P) = \ell(nP) + 1$.

7. A non-singular projective curve X is covered by two affine pieces (with respect to different embeddings) which are affine plane curves with equations $y^2 = f(x)$ and $v^2 = g(u)$ respectively, with f a square-free polynomial of even degree $2n$ and $u = 1/x$, $v = y/x^n$ in $K(X)$. Determine the polynomial $g(u)$ and show that the canonical class on X has degree $2n - 4$. Why can we not just say that X is the projective plane curve associated to the affine curve $y^2 = f(x)$?

8. Let $X_0 \subset \mathbb{A}^2$ be the affine curve with equation $y^3 = x^4 + 1$, and let $X \subset \mathbb{P}^2$ be its projective closure. Show that X is smooth, and has a unique point Q at infinity. Let ω be the rational differential dx/y^2 on X. Show that $v_P(\omega) = 0$ for all $P \in X_0$. prove that $v_Q(\omega) = 4$ and hence that ω, $x\omega$ and $y\omega$ are all regular on X.

9. Let X be a non-singular projective curve and $P \in X$ any point. Show that there exists a nonconstant rational function on X which is regular everywhere except at P. Show moreover that there exists a projective embedding of X which has P as its unique point at infinity.

10. Let P_0 be a point on an elliptic curve (non-singular projective curve of genus 1!) and $\phi_{3P_0}\colon X \to \mathbb{P}^2$ the projective embedding. Show that $P \in X$ is a point of inflection if and only if $3P = 0$ in the group law determined by P_0. Deduce that if P and Q are points of inflection then so is the third point of intersection of the line PQ with X.

11. Let X be a projective non-singular curve of genus 2, and let K be an effective canonical divsor on X. Consider the divisor $K + P_1 + P_2$ for points P_i with $P_1 + P_2 \not\sim K$. Show that the linear system associated to this divisor induces a morphism ϕ from X to a quartic curve in $\mathbb{P}^2$. Show furthermore that $\phi(P) = \phi(Q)$, with $P \neq Q$, if and only if $\{P, Q\} = \{P_1, P_2\}$.

12. Let $\pi\colon X \to \mathbb{P}^1$ be a hyperelliptic cover, i.e., a non-constant map of degree 2 from the projective non-singular curve X, and let $P \neq Q$ be ramification points of π. Show that as elements of $Cl^0(X)$, $P - Q \neq 0$ but $2(P - Q) = 0$.