TORIC GEOMETRY, SHEET I: MICHAELMAS 2019

1. Given a cone σ in $N_{\mathbb{R}}$ prove that the double dual $(\sigma^\vee)^\vee$ is canonically identified with σ.
2. Consider the 3 cones in $\mathbb{R}^2$ given generated (1) by $(0,1)$ and $(1,1)$, (2) by $(1,1)$, and (3) by $(1,0)$ and $(1,1)$. Draw the respective dual cones. These three cones assemble to form a fan. Check by hand in this case that the resulting variety is separated.
3. Give an explicit construction of the toric variety $\mathbb{P}^2 \times \mathbb{P}^1$ from a fan – you should describe each affine toric open subsets, the gluing morphisms, and why the resulting toric variety is $\mathbb{P}^2 \times \mathbb{P}^1$.
4. Let σ be a cone in $N_{\mathbb{R}}$ and assume it is full dimensional, i.e. the span of σ is $N_{\mathbb{R}}$. Recall that by choosing generators for S_σ, we may regard U_σ as a subset of $\mathbb{C}^N$ and endow it with the Euclidean topology, obtaining a topological space U_σ^{an}. Show that U_σ^{an} is contractible as a topological space.
5. Consider the fan Σ in $\mathbb{R}^2$ whose rays are generated by $(1,0)$, $(0,1)$ and $(-1,-1)$ and which does not have any 2-dimensional cones. Describe the toric variety X_Σ explicitly. Calculate the Euler characteristic[1] of the topological space X_Σ^{an} .
6. Give an example of a toric variety X of dimension 3 (*a toric threefold*) and a point $x \in X$ such that the tangent space T_xX at this point has dimension 4. Are there any restrictions on the dimension of the tangent space of a point on a toric threefold?
7. Let Σ and Σ' be fans in vector spaces $N_{\mathbb{R}}$ and $N'_{\mathbb{R}}$. Show that the product $\Sigma \times \Sigma'$ naturally has the structure of a fan in $N_{\mathbb{R}} \oplus N_{\mathbb{R}}$ and therefore defines a toric variety. Moreover, show that there is an isomorphism
$$X_{\Sigma\times\Sigma'} \cong X_\Sigma \times X_{\Sigma'},$$
so *the construction of a toric variety from a fan commutes with products.*
8. Let X be a toric variety with dense torus T. Recall that we partitioned the cocharacter lattice N of T based on the limits of one parameter subgroups of T inside X (this will turn out to be the fan attached to X). Give an example to show that this data does not uniquely determine X.
9. (Hirzebruch surfaces) Let Σ_r be the fan in $\mathbb{R}^2$ whose rays are given by
$$(1,0),(0,1),(-1,r),(0,-1)$$
for r a non-negative integer, and whose two-dimensional cones are spanned by adjacent pairs of vectors, i.e.
$$\langle(1,0),(0,1)\rangle, \langle(0,1),(-1,r)\rangle, \langle(-1,r),(0,-1)\rangle, \langle(0,-1),(1,0)\rangle$$

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[1]If you are confused by this, first calculate the Euler characteristic of $\mathbb{P}^2$ by giving it a cell decomposition.

Show that $X_r = X_{\Sigma_r}$ admits a morphism to $\mathbb{P}^1$ such that every fiber is isomorphic to $\mathbb{P}^1$, so X_r is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. Show also that X_1 is not isomorphic to X_0.

10. Give an example of a morphism of smooth toric varieties $X \to \mathbb{C}^2$ such that the fiber over $(0,0)$ in $\mathbb{C}^2$ is singular.
11. In lecture, we claimed that the toric variety U_σ associated to a cone σ is smooth if and only if it is generated by a subset of a $\mathbb{Z}$-basis for N. Give a complete proof of this statement. Give an example to show that if σ is generated by a subset of a $\mathbb{Q}$-basis then U_σ need not be smooth.
12. Recall that $\mathcal{O}_{\mathbb{P}^1}(1)$ is a line bundle on $\mathbb{P}^1$ whose fiber over a point is the line in $\mathbb{C}^2$ associated to that point. Prove that the total space of this bundle $\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(1))$ is a 2-dimensional toric variety[2].

($\star$) Prove that there is an open embedding of $\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(1))$ into the Hirzebruch surface X_1. We will see later that X_r are all compact so X_1 is a compactification of the total space of this bundle.

[2] If you feel you don't have enough background for this question, go look at the Wikipedia definition for a line bundle using local trivializations. Once you're done with that, you'll have enough background!